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HAMILTONIAN FORMULATION AND FUNDAMENTAL CONSERVATION LAWS FOR A MODEL OF SMALL ELLIPTICAL VORTICES*

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A Hamiltonian formulation and fundamental conservation laws are considered for a model of small elliptical vortices /1/ which, while maintaining the attractive aspects of Kirchhoff's point-vortex model, makes approximate allowance for the internal dynamics of vortices. As an example, the motion of two vortices in the direction of the normal to a straight line through their centres is considered.

When coherent structures in the two-dimensional flow of an ideal incompressible fluid are modelled by Kirchhoff point vortices, neither the internal structure of the vortices nor the changes in their shapes are taken into consideration. For example, the "vortex pairing" phenomenon observed in shear flows /2/ cannot be described in the context of the point-vortex model. Several authors /3, 4/ have represented vortices as finite regions of constant vorticity. The shape of the vortices is not known in advance but determined by a numerical solution of Euler's equations by the contour dynamical method /3/. As the method is cumbersome, it is useful only when the number of vortices is small.

The internal dynamics of vortices may be taken into account by considering elliptically shaped vortices, within which the vorticity is assumed to be constant /1/. This constant vorticity condition is not obligatory. One can also consider other vorticity distributions, such as hollow vortices, in which the vorticity is concentrated on the boundary. However, the constant vorticity model is physically reasonable in view of the Prandtl-Batchelor Theorem /5/, if one is interested in applications to the modelling of flows at large Reynolds numbers. The assumption that the vortices are elliptical is also natural for the following reasons. First, as Kirchhoff showed /5/, an isolated elliptical vortex rotates at constant angular velocity without changing shape. Second, the same vortex in an external flow with stream function

$$\psi = \text{const}\, xy \tag{1}$$

remains elliptical in motion, though the ratio of its axes may vary /6/.

In a system of small elliptical vortices, in which the characteristic distance between vortices may exceed their characteristic dimension, the internal dynamics of such vortex splits into a Kirchhoff rotation and a motion driven by the external field due to the other vortices. The latter has the form (1) in the principal term, so that such vortices remain elliptical.

Consider a system of N elliptical vortices in an infinite ideal incompressible fluid. Let ω_α be the vorticity of the α-th vortex, a_α, b_α its **semi-axes,** $A_\alpha = \pi a_\alpha b_\alpha$ its area, $\Gamma_\alpha = A_\alpha \omega_\alpha$ the circulation, $\lambda_\alpha = a_\alpha / b_\alpha$, $f_\alpha = \lambda_\alpha + (\lambda_\alpha)^{-1}$, φ_α the angle between the axis of the ellipse and the x axis, $z_\alpha = (x_\alpha, y_\alpha)$ the position of the vortex centre. The relative position of a pair of vortices is described by polar coordinates $R_{\alpha\beta}$, $\theta_{\alpha\beta}$, so that $z_\alpha - z_\beta = R_{\alpha\beta}(\cos\theta_{\alpha\beta}, \sin\theta_{\alpha\beta})$.

In the case considered here - a piecewise-constant distribution of vorticity - the energy of the motion may be expressed as

$$H = -\frac{1}{4\pi} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \omega_\alpha \omega_\beta J_{\alpha\beta}, \quad J_{\alpha\beta} = \iint \ln|z - \zeta|\, dz\, d\zeta \tag{2}$$

where the integration is performed over the α-th and β-th ellipses.

We first consider $J_{\alpha\beta}$ for $\alpha \neq \beta$. Writing $z = z_\alpha + z'$, $\zeta = \zeta_\beta + \zeta'$ and noting that, by virtue of our initial assumptions, $|z'|, |\zeta'| \ll R_{\alpha\beta}$, we expand $\ln|z - \zeta|$ up to second-order terms:

$$\ln|z - \zeta| = \ln R_{\alpha\beta} + \frac{(z' - \zeta')(z_\alpha - z_\beta)}{R_{\alpha\beta}^2} + \left\{\frac{|z' - \zeta'|^2}{2R_{\alpha\beta}^2} - \frac{[(z' - \zeta')(z_\alpha - z_\beta)]^2}{R_{\alpha\beta}^4}\right\} + \ldots$$

After integration, the first term will give the usual Kirchhoff Hamiltonian, while the contribution of the second vanishes owing to symmetry. Integration of the third term again utilizes symmetry considerations, and the following integrals are evaluated:

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$$\int |z'|^2 dz' = \frac{A^2}{4\pi} f, \quad \iint \frac{[(z'-\zeta')(z_\alpha - z_\beta)]^2}{R^2_{\alpha\beta}} dz'\, d\zeta' =$$
$$\tfrac{1}{8} A_\alpha A_\beta [a_\alpha^2 + b_\alpha^2 + a_\beta^2 + b_\beta^2 + (a_\alpha^2 - b_\alpha^2)\cos(2\varphi_\alpha - 2\theta_{\alpha\beta}) + (a_\beta^2 - b_\beta^2)\cos(2\varphi_\beta - 2\theta_{\alpha\beta})]$$

Unlike $J_{\alpha\beta}$, $J_{\alpha\alpha}$ must be evaluated exactly. Cumbersome calculations yield

$$J_{\alpha\alpha} = A_\alpha^2 K(f_\alpha), \quad K(f) = \frac{1}{2}\ln\left(\frac{f+2}{4\pi}\right) - \frac{1}{4}$$

The final formula for the Hamiltonian is

$$H = -\frac{1}{2\pi}\sum_{\alpha<\beta}\sum \Gamma_\alpha\Gamma_\beta \ln R_{\alpha\beta} - \frac{1}{4\pi}\sum_{\alpha=1}^{N}\Gamma_\alpha^2 K(f_\alpha) + \sum_{\alpha<\beta}\sum \frac{\Gamma_\alpha\Gamma_\beta}{16\pi R^2_{\alpha\beta}}[(a_\alpha^2 - b_\alpha^2)\cos(2\theta_{\alpha\beta} - 2\varphi_\alpha) + (a_\beta^2 - b_\beta^2)\cos(2\theta_{\alpha\beta} - 2\varphi_\beta)] \quad (3)$$

The equations for the trajectories of the fluid particles are found from the usual formulae

$$\omega x^{\cdot} = \delta H/\delta y, \quad \omega y^{\cdot} = -\delta H/\delta x \quad (4)$$

where $\delta/\delta x$, $\delta/\delta y$ are variational derivatives. Using the explicit formula for the Hamiltonian (3), one can reduce the infinite-dimensional Hamiltonian system (4) to the finite-dimensional system

$$\Gamma_\alpha x_\alpha^{\cdot} = \partial H/\partial y_\alpha, \quad \Gamma_\alpha y_\alpha^{\cdot} = -\partial H/\partial x_\alpha \quad (5)$$
$$(8\pi)^{-1}\Gamma_\alpha A_\alpha f_\alpha^{\cdot} = \partial H/\partial\varphi_\alpha, \quad (8\pi)^{-1}\Gamma_\alpha A_\alpha \varphi_\alpha^{\cdot} = -\partial H/\partial f_\alpha$$

Thus, the system of small elliptical vortices is described by the Hamiltonian Eqs.(5) with $2N$ degrees of freedom, relative to which (x_α, y_α) and $(f_\alpha, \varphi_\alpha)$ are conjugate canonical variables. The first two equations of (5) are an immediate **generalization of** Kirchhoff's Eqs. /5/ and describe the external dynamics of the vortices, while the other equations describe the internal dynamics. Invariance under translations and rotations of the coordinate frame yields the integrals of motion

$$J_1 = \Sigma\Gamma_\alpha x_\alpha, \quad J_2 = \Sigma\Gamma_\alpha y_\alpha \quad (6)$$
$$J_3 = \Sigma\Gamma_\alpha [x_\alpha^2 + y_\alpha^2 + (4\pi)^{-1}A_\alpha f_\alpha]$$

It was assumed previously that the characteristic dimension of the vortex is of the order of $\sqrt{A}$. However, situations may occur in which the vortices are strongly stretched, so that the characteristic dimension should be the major **semi-axis.** For example, it has been shown /6/ that a single elliptical vortex in a shear flow may stretch to infinity. It is therefore interesting to consider the limiting model of small elliptical vortices - a model of vortical segments, in which a_α is assumed to be finite, $b_\alpha \to 0$, $\omega_\alpha \to \infty$, so that the circulation Γ_α, $\pi a_\alpha b_\alpha \omega_\alpha$ remains finite. The condition that the vortices be small is now $a \ll R$.

In this limiting case, one must put $b_\alpha = b_\beta = 0$, $K(f_\alpha) = \ln a_\alpha$ in formula (3) for the Hamiltonian, and the last two equations for motion (5) are replaced by

$$(\Gamma_\alpha/8)(a_\alpha^2)^{\cdot} = \partial H/\partial\varphi_\alpha, \quad (\Gamma_\alpha/8)\varphi_\alpha^{\cdot} = -\partial H/\partial a_\alpha^2$$

It is clear that the conjugate canonical variables are (x_α, y_α) and $(a_\alpha^2, \varphi_\alpha)$. Of the integrals of motion (6), only the last one must be changed, replacing the quantity $(4\pi)^{-1}A_\alpha f_\alpha$ by $a_\alpha^2/4$.

Examples of the motion of small elliptical vortices. Consider the motion of two vortices. This is a Hamiltonian system with four degrees of freedom, and by Liouville's Theorem /7/, a necessary condition for its integrability is the existence of three additional integrals of motion, which are in involution with respect to Poisson brackets. It is easy to show that the Poisson brackets associated with the Hamiltonian Eqs.(5), for two arbitrary functions F, G, are given by

$$\{F, G\} = \Sigma\left[\frac{1}{\Gamma_\alpha}\frac{\partial(F,G)}{\partial(x_\alpha, y_\alpha)} + \frac{8\pi}{\Gamma_\alpha A_\alpha}\frac{\partial(F,G)}{\partial(f_\alpha, \varphi_\alpha)}\right]$$

The three additional integrals of (6) are not in involution:

$$\{J_1, J_2\} = \Sigma\Gamma_\alpha, \quad \{J_1, J_3\} = 2J_2, \quad \{J_2, J_3\} = -2J_1$$

Two integrals of motion which are in involution can be constructed, e.g., J_3 and $J_1^2 + J_2^2$. We cannot prove that there is no third independent integral of motion of involution with them. In an analogous, simpler situation, with Kirchhoff vortices, numerical results /8, 9/ indicate that in general even a system of two small elliptical vortices is stochastic. However, if the system admits of additional discrete symmetry, it may turn out to be integrable.

As an example, consider the motion of a pair of vortices symmetrical about the x axis: a vortex with circulation Γ and coordinates (x, y, f, φ) and another with circulation Γ and coordinates $(x, -y, f, -\varphi)$. The first and third conservation laws (6) are identically valid, while the second yields $y = y_0 = \text{const}$. The energy conservation law yields yet another integral of motion

$$\varepsilon \frac{1-\lambda^2}{\lambda} \cos 2\varphi + \ln (f+2) = \text{const} \left(\varepsilon = \frac{A}{2\pi R^2} = \frac{A}{8\pi y_0^2} = \text{const}\right) \tag{7}$$

The last two equations of motion (5) yield

$$\lambda^{\cdot} = -2\varepsilon\omega\lambda \sin 2\varphi, \quad \varphi^{\cdot} = \frac{\lambda\omega}{(1+\lambda)^2} + \frac{1+\lambda^2}{1-\lambda^2} \varepsilon\omega \cos 2\varphi \tag{8}$$

Note that the conservation law (7) can be derived directly from Eq.(8).

The longitudinal coordinate of the vortices is found from the first two of Eqs.(5):

$$x^{\cdot} = -\frac{\Gamma}{2\pi R} \left[1 + 2\varepsilon \frac{1-\lambda^2}{\lambda} \cos 2\varphi\right] \tag{9}$$

Eqs.(8) and (9) are readily integrated numerically. However, as the results presented above hold for $\varepsilon \ll 1$, we will use perturbation theory, seeking a solution in the form

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + O(\varepsilon^2), \quad \lambda = \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^2)$$
$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

Substituting this expansion into (8), (9) and assuming for simplicity that $\lambda_0 \neq 1$, we finally obtain

$$\varphi = \Omega t + \varepsilon\omega \left(\frac{\lambda_0}{1-\lambda_0^2} \frac{\sin 2\Omega t}{\Omega} - \frac{1-\lambda_0}{1+\lambda_0} t\right) + O(\varepsilon^2) \tag{10}$$
$$\lambda = \lambda_0 + \varepsilon\lambda_0\omega \frac{\cos 2\Omega t - 1}{\Omega} + O(\varepsilon^2)$$
$$x = -\frac{\Gamma}{2\pi R} \left[t + \varepsilon \frac{1-\lambda_0^2}{\lambda_0} \frac{\sin 2\Omega t}{\Omega}\right] + O(\varepsilon^2); \quad \Omega = \frac{\lambda_0\omega}{(1+\lambda_0)^2}$$

Here Ω is the angular velocity of rotation of a single vortex; the initial conditions are $\lambda(0) = \lambda_0$, $x(0) = 0$, $\varphi(0) = 0$. It is obvious that the secondary motion of the order of ε is periodic and its frequency is double that of the fundamental motion. Terms of the order of ε^2 are disregarded, since they were not taken into account in the fundamental equations (5).

A detailed investigation of the motion of two small vortices was undertaken in /10, 11/.

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